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THE GENERALIZED HYERS-ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATION WITH AN INVOLUTION IN NON-ARCHIMEDEAN SPACES

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Abstract. In this paper, using fixed point method, we prove the Hyers-Ulam stability of the following functional equation

$(k+1)f(x+y)+f(x+\sigma(y))+kf(\sigma(x)+y)-2(k+1)f(x)-2(k+1)f(y)=0$

with an involution σ for a fixed non-zero real number k with $k \neq -1$.

1. Introduction and preliminaries

In 1940, Ulam [13] posed the following problem concerning the stability of functional equations: *Let G_1 be a group and let G_2 a meric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

Hyers [6] solved the Ulam's problem for the case of approximately additive functions in Banach spaces. Since then, the stability of several functional equations have been extensively investigated by several mathematicians [3, 5, 7, 8]. The Hyers-Ulam stability for the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

was proved by Skof [11] for a function $f : E_1 \longrightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space and later by Jung [10] on unbounded domains.

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Let X and Y be real vector spaces. For an additive mapping $\sigma : X \longrightarrow X$ with $\sigma(\sigma(x)) = x$ for all $x \in X$, then σ is called *an involution* of X. For a given involution $\sigma : X \longrightarrow X$, the functional equation

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) \tag{1.2}$$

for all $x, y \in X$ is called *the quadratic functional equation with an involution* and a solution of (1.2) is called *a quadratic mapping with an involution.* The functional equation (1.2) has been studied by Stetkær [12] and the Hyers-Ulam-Rassias Theorem has been obtained by Bouikhalene et al. [1, 2, 9].

In this paper, using fixed point method, we prove the generalized Hyers-Ulam stability of the following functional equation

$$(k+1)f(x+y)+f(x+\sigma(y))+kf(\sigma(x)+y)-2(k+1)f(x)-2(k+1)f(y) = 0 \tag{1.3}$$

for a fixed non-zero real number k with $k \neq -1$.

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that for any $r, s \in K$, the following conditions hold: (i) $|r| = 0$ if and only if $r = 0$, (ii) $|rs| = |r||s|$, and (iii) $|r + s| \leq |r| + |s|$.

A field K is called *a valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. If the triangle inequality is replaced by $|r + s| \leq max\{|r|, |s|\}$ for all $r, s \in K$, then the valuation $|\cdot|$ is called a *non-Archimedean valuation* and the field with a non-Archimedean valuation is called *non-Archimedean field.* If $|\cdot|$ is a non-Archimedean valuation on K, then clearly, $|1| = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Definition 1.1. Let X be a vector space over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is called *a non-Archimedean norm* if satisfies the following conditions:
(a) $\|x\| = 0$ if and only if $x = 0$,
(b) $\|rx\| = |r|\|x\|$, and
(c) the strong triangle inequality (ultrametric) holds, that is,

$$||x + y|| \leq max\{\|x\|, \|y\|\}$$

for all $x, y \in X$ and all $r \in K$.

If $\|\cdot\|$ is a non-Archimedean norm, then $(X, \|\cdot\|)$ is called *a non-Archimedean normed space.* Let $(X, \|\cdot\|)$ be a non-Archimedean normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. In that case, x is called *the limit of the sequence* $\{x_n\}$, and one denotes it by $\lim_{n\to\infty} x_n =$

x. A sequence $\{x_n\}$ is said to be *a Cauchy sequence* if $\lim_{n\to\infty} \|x_{n+p} - x_n\| = 0$ for all $p \in \mathbb{N}$. Since

$$\|x_n - x_m\| \le max\{\|x_{j+1} - x_j\| \mid m \le j \le n-1\} \quad (n > m),$$

a sequence $\{x_n\}$ is Cauchy in $(X, \|\cdot\|)$ if and only if $\{x_{n+1}-x_n\}$ converges to zero in $(X, \|\cdot\|)$. By *a complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

THEOREM 1.2. [4] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point x^* of J ;*
(3) *x^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$* and
(4) *$d(y, y^*) \le \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, we assume that X is a non-Archimedean normed space and Y is a complete non-Archimedean normed space.

2. Solutions of (1.3)

In this section, we investigate solutions of (1.3). We start with the following lemma.

LEMMA 2.1. *Let $f : X \longrightarrow Y$ be mapping. Then f satisfies* (1.3) *if and only if f is a quadratic mapping with an involution.*

Proof. Suppose that f satisfies (1.3). Letting $x = y = 0$ in (1.3), we have $f(0) = 0$. Letting $x = x + \sigma(x),\ y = x + \sigma(x)$ in (1.3), we have

$$2(k+1)f(2(x+\sigma(x))) = 4(k+1)f(x+\sigma(x))$$

for all $x \in X$ and since $k \neq -1$,

$$f(2(x+\sigma(x))) = 2f(x+\sigma(x)) \tag{2.1}$$

for all $x \in X$. Letting $x = x + \sigma(y),\ y = \sigma(x) + y$ in (1.3), we get

$$\begin{aligned}(k+1)f(x+y+\sigma(x+y)) + f(2(x+\sigma(y))) + kf(2(\sigma(x)+y))\\ = 2(k+1)f(x+\sigma(y)) + 2(k+1)f(\sigma(x)+y))\end{aligned} \tag{2.2}$$

for all $x, y \in X$. Letting $x = \sigma(x) + y,\ y = x + \sigma(y)$ in (1.3), we get

$$\begin{aligned}(2.3)\quad &(k+1)f(x+y+\sigma(x+y)) + f(2(\sigma(x)+y)) + kf(2(x+\sigma(y)))\\ &= 2(k+1)f(\sigma(x)+y) + 2(k+1)f(x+\sigma(y)))\end{aligned}$$

for all $x, y \in X$. From (2.2) and (2.3),

$$f(2(x+\sigma(y))) = f(2(\sigma(x)+y))$$

for all $x, y \in X$. By (2.1),

$$f(x+\sigma(y)) = f(\sigma(x)+y) \tag{2.4}$$

for all $x, y \in X$. Substituting (2.4) by (1.3), we get

$$f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y)$$

Therefore f be quadratic with an involution.

Assume that $f(x+y)+f(x+\sigma(y)) = 2f(x)+2f(y)$. Letting $x = y = 0$ in (1.3), we have $f(0) = 0$. Letting $x = 0$ in (1.3), we have

$$f(y) = f(\sigma(y)) \tag{2.5}$$

for all $y \in X$. By (2.5)

$$f(x+y) + f(\sigma(x)+y) = 2f(x) + 2f(y) \tag{2.6}$$

for all $x, y \in X$. From (1.3) and (2.6), f satisfies (1.3). □

REMARK 2.2. The mapping $f : X \longrightarrow Y$ satisfying (1.3) for the case $k = -1$ is not quadratic. In fact, for $a \in Y$ with $a \neq 0$, the constant mapping $f(x) = a$ satisfies (1.3) but it is not quadratic.

3. The generalized Hyers-Ulam stability for (1.3)

Using the fixed point methods, we will prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.3) with an involution σ in non-Archimedean normed space.

THEOREM 3.1. *Assume that $\phi : X^2 \longrightarrow [0, \infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\phi(2x, 2y) \leq |4|L\phi(x, y),\ \ \phi(x+\sigma(x), y+\sigma(y)) \leq |4|L\phi(x, y) \tag{3.1}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\begin{aligned}&\|(k+1)f(x+y) + f(x+\sigma(y))\\ &\quad + kf(\sigma(x)+y) - 2(k+1)f(x) - 2(k+1)f(y)\| \leq \phi(x, y)\end{aligned} \tag{3.2}$$

for all $x, y \in X$ and a fixed real number k with $k \neq -1$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ with an involution such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4(k+1)|(1-L)}\phi(x,x) \tag{3.3}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by $d(g,h) = inf\{c \in [0,\infty) | \ \|g(x) - h(x)\| \leq c\, \phi(x,x)$ for all $x \in X\}$. Then (S,d) is a complete metric space(See [9]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = \frac{1}{4}\{g(2x)+g(x+\sigma(x))\}$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some non-negative real number c. Then by (3.1), we have

$$\begin{aligned}
&\|Jg(x) - Jh(x)\| \\
&= \frac{1}{|4|}\ \|g(2x) + g(x+\sigma(x)) - h(2x) - h(x+\sigma(x))\| \\
&\leq \frac{1}{|4|} max\{\|g(2x) - h(2x)\|, \|g(x+\sigma(x)) - h(x+\sigma(x))\|\} \\
&\leq cL\phi(x,x)
\end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Next, we claim that $d(Jf, f) < \infty$. Putting $y = x$ in (3.2) and dividing both sides by $|4(k+1)|$, we get

$$\left\|\frac{1}{4}\{f(2x) + f(x+\sigma(x))\} - f(x)\right\| = \|Jf(x) - f(x)\| \leq \frac{1}{|4(k+1)|}\phi(x,x)$$

for all $x \in X$ and hence

$$d(Jf, f) \leq \frac{1}{|4(k+1)|} < \infty. \tag{3.4}$$

By Theorem 1.2, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, Q) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n f)(x) = \frac{1}{2^{2n}}\Big\{f(2^n x) + (2^n - 1)f\Big(2^{n-1}\big(x+\sigma(x)\big)\Big)\Big\}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $d(J^n f, Q) \to 0$ as $n \to \infty$, there exists a sequence $\{c_n\}$ in $\mathbb{R}$ such that $c_n \to 0$ as $n \to \infty$ and $d(J^n f, Q) \leq c_n$ for every $n \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\|(J^n f)(x) - Q(x)\| \leq c_n \phi(x,x)$$

for all $x \in X$. Thus for each fixed $x \in X$, we have

$$\lim_{n\to\infty} ||(J^n f)(x) - Q(x)|| = 0$$

and

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}\Big\{f(2^n x) + (2^n - 1)f\Big(2^{n-1}\big(x + \sigma(x)\big)\Big)\Big\}. \tag{3.5}$$

It follows from (3.2) and (3.5) that

$$\|(k+1)Q(x+y)+Q(x+\sigma(y))+kQ(\sigma(x)+y)-2(k+1)Q(x)-2(k+1)Q(y)\|$$

$$\leq \lim_{n\to\infty} \frac{1}{|4|^n}\ max\{\phi(2^n x, 2^n y), |2^n - 1|\phi(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y)))\}$$

$$\leq \lim_{n\to\infty} \frac{1}{|4|^n}\ max\{|4|^n L^n \phi(x, y), |2^n - 1||4|^n L^n \phi(x, y)\} = 0$$

for all $x, y \in X$, because $|4|L < 1$ and $|2^n - 1| < 1$. Hence Q satisfies (1.3) and by Lemma 2.1, Q is a quadratic mapping with an involution. By (4) in Theorem 1.2 and (3.4), f satisfies (3.3).

Assume that $Q_1 : X \longrightarrow Y$ is another solution of (1.3) satisfying (3.3). We know that Q_1 is a fixed point of J. Due to (3) in Theorem 1.2, we get $Q = Q_1$. This proves the uniqueness of Q. □

Theorem 3.2. *Assume that* $\phi : X^2 \longrightarrow [0, \infty)$ *is a mapping and there exists a real number* L *with* $0 < L < 1$ *such that*

$$\phi(x, y) \leq \frac{L}{|4|}\phi(2x, 2y),\ \ \phi(x + \sigma(x), y + \sigma(y)) \leq \frac{L}{|2|}\phi(4x, 4y) \tag{3.6}$$

for all $x, y \in X$. *Let* $f : X \longrightarrow Y$ *be a mapping satisfying* (3.2) *and* $f(0) = 0$. *Then there exists a unique quadratic mapping* $Q : X \longrightarrow Y$ *with an involution such that*

$$\|f(x) - Q(x)\| \leq \frac{L}{|4(k + 1)|(1 - L)}\phi(x, x) \tag{3.7}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by $d(g, h) = inf\{c \in [0, \infty)|\ \|g(x) - h(x)\| \leq c\ \phi(x, x)$ for all $x \in X\}$. Then (S, d) is a complete metric space. Define a mapping $J : S \longrightarrow S$ by

$$Jg(x) = 4\Big\{g\Big(\frac{x}{2}\Big) - \frac{1}{2}g\Big(\frac{x + \sigma(x)}{4}\Big)\Big\}$$

for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some non-negative real number c. Then by (3.13), we have

$$\begin{aligned}
&\|Jg(x) - Jh(x)\| \\
&= |4|\Big\| g\Big(\frac{x}{2}\Big) - \frac{1}{2}g\Big(\frac{x+\sigma(x)}{4}\Big) - h\Big(\frac{x}{2}\Big) + \frac{1}{2}h\Big(\frac{x+\sigma(x)}{4}\Big)\Big\| \\
&\leq |4| max\Big\{\Big\| g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big)\Big\|, \frac{1}{|2|}\Big\| g\Big(\frac{x+\sigma(x)}{4}\Big) - h\Big(\frac{x+\sigma(x)}{4}\Big)\Big\|\Big\} \\
&\leq cL\phi(x, x)
\end{aligned}$$

for all $x \in X$. Hence $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Next, we claim that $d(Jf, f) < \infty$. Putting $x = \frac{x}{2}$ and $y = \frac{x}{2}$ in (3.2) and dividing both sides by $|k + 1|$, we get

$$\Big\| f(x) + f\Big(\frac{x+\sigma(x)}{2}\Big) - 4f\Big(\frac{x}{2}\Big)\Big\| \leq \frac{1}{|k+1|}\phi\Big(\frac{x}{2}, \frac{x}{2}\Big) \tag{3.8}$$

and putting $x = \frac{x+\sigma(x)}{4}$ and $y = \frac{x+\sigma(x)}{4}$ in (3.2) and dividing both sides by $|2(k + 1)|$, we get

$$\Big\| f\Big(\frac{x+\sigma(x)}{2}\Big) - 2f\Big(\frac{x+\sigma(x)}{4}\Big)\Big\| \leq \frac{1}{|2(k+1)|}\phi\Big(\frac{x+\sigma(x)}{4}, \frac{x+\sigma(x)}{4}\Big) \tag{3.9}$$

for all $x \in X$. Combining (3.8) and (3.9), by (3.6), we deduce that

$$\|Jf(x) - f(x)\| = \Big\| 4f\Big(\frac{x}{2}\Big) - 2f\Big(\frac{x+\sigma(x)}{4}\Big) - f(x)\Big\| \leq \frac{L}{|4(k+1)|}\phi(x, x)$$

for all $x \in X$ and hence

$$d(Jf, f) \leq \frac{L}{|4(k+1)|} < \infty.$$

By Theorem 1.2, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, Q) \to 0$ as $n \to \infty$. By induction, we can easily show that

$$(J^n f)(x) = 2^{2n}\Big\{ f\Big(\frac{x}{2^n}\Big) + \Big(\frac{1}{2^n} - 1\Big) f\Big(\frac{x+\sigma(x)}{2^{n+1}}\Big)\Big\}$$

for each $n \in \mathbb{N}$. Since $d(J^n f, Q) \to 0$ as $n \to \infty$, there exists a sequence $\{c_n\}$ in $\mathbb{R}$ such that $c_n \to 0$ as $n \to \infty$ and $d(J^n f, Q) \leq c_n$ for every $n \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\|(J^n f)(x) - Q(x)\| \leq c_n\phi(x, x)$$

for all $x \in X$. Thus for each fixed $x \in X$, we have

$$\lim_{n\to\infty} \|(J^n f)(x) - Q(x)\| = 0$$

and

$$Q(x) = 2^{2n}\Big\{f\Big(\frac{x}{2^n}\Big) + \Big(\frac{1}{2^n} - 1\Big)f\Big(\frac{x+\sigma(x)}{2^{n+1}}\Big)\Big\}.$$

Analogously to the proof of Theorem 3.1, we can show that Q is a unique quadratic mapping with an involution satisfying (3.7). $\square$

As examle of $\phi(x, y)$ in Theorem 3.1 and Theorem 3.2, we can take $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we have the following corollary.

COROLLARY 3.3. *Let $\theta \geq 0$ and p be a positive real number with $p \neq 2$. Suppose that $|2| < 1$. Let $f : X \longrightarrow Y$ be a mapping satisfying*

(3.10)

$$\begin{aligned} &\|(k+1)f(x+y) + f(x+\sigma(y)) \\ &\quad + kf(\sigma(x)+y) - 2(k+1)f(x) - 2(k+1)f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \end{aligned}$$

and $\|x + \sigma(x)\|^p \leq |2|^p\|x\|^p$ for all $x, y \in X$. Then there exists a unique mapping $Q : X \longrightarrow Y$ with an involution such that Q is a solution of the functional equation (1.3) *and the inequality*

$$||f(x) - Q(x)|| \leq \begin{cases} \dfrac{2}{|1+k|(|4| - |2|^p)}\theta||x||^p, & \text{if } p > 2, \\[2ex] \dfrac{2}{|1+k|(|2|^p - |4|)}\theta||x||^p, & \text{if } 0 \leq p < 2 \end{cases}$$

holds for all $x \in X$.

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